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1991 J. Phys. A: Math. Gen. 24 2599

(http://iopscience.iop.org/0305-4470/24/11/026)

J. Phys. A: Math. Gen. 24 (1991) 2599–2614. Printed in the UK

Temperley–Lieb lattice models arising from quantum groups

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Abstract. We construct families of solvable quantum spin chains, vertex and interactions round a face (IRF) models from representations of the Temperley–Lieb algebra associated with the quantum groups $\mathcal{U}_q(X_n)$ for $X_n = A_1, B_n, C_n$ and D_n. We determine a correlation measuring order in the IRF models via the fusion rule in the level l $X_n^{(1)}$ Wess–Zumino–Witten model.

1. Introduction

The advent of quantum groups has lead to considerable progress in the field of solvable lattice models in two-dimensional statistical mechanics. It provides a powerful algebraic framework for constructing and analysing various kinds of models. Another such structure notable in the theory of solvable models is the Temperley–Lieb (TL) algebra [1]:

$$U_j^2 = \sqrt{Q}U_j \tag{1.1a}$$

$$U_j U_{j\pm1} U_j = U_j \tag{1.1b}$$

$$U_i U_j = U_j U_i \qquad |i-j| > 1. \tag{1.1c}$$

This algebra appears in a large class of solvable models and is known to essentially govern their physical properties. The association of the TL algebra leads to some equivalence relations among the models as formulated in [2].

In this paper we study solvable lattice models associated with the TL algebra in the framework of quantum groups. We start from a finite-dimensional irreducible representation $\pi : \mathcal{U}_q \to \mathrm{End}\, V_{\bar{\xi}}$ of a quantized universal enveloping algebra $\mathcal{U}_q$ with highest weight $\bar{\xi}$ such that the decomposition $V_{\bar{\xi}} \otimes V_{\bar{\xi}}$ is multiplicity free and includes *one trivial representation on* V_0 *(dim* $V_0 = 1$*). Given such* $\mathcal{U}_q$ and π, *representations of* the TL algebra can be constructed by using the projector onto the space V_0 with the value of Q given by

$$\sqrt{Q} = \mathrm{Tr}_{V_{\bar{\xi}}}(q^{-2\bar{\rho}}) \tag{1.2}$$

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0305-4470/91/112599+16$03.50

where $\bar{\rho}$ is the half-sum of positive roots of $\mathcal{U}_q$ [3].

As concrete examples we consider the cases $(V_{\bar{\xi}}, \mathcal{U}_q(X_n)) = (V_{2s\bar{\Lambda}_1}, \mathcal{U}_q(A_1))$ for spin-s, $(V_{\bar{\Lambda}_1}, \mathcal{U}_q(B_n))$, $(V_{\bar{\Lambda}_1}, \mathcal{U}_q(C_n))$ and $(V_{\bar{\Lambda}_1}, \mathcal{U}_q(D_n))$. Namely we treat the q-deformations of the spin-s representation of sl(2) and the vector representation of so$(2n+1)$, sp$(2n)$ and so$(2n)$. Here, for example, $V_{2s\bar{\Lambda}_1}$ denotes the $\mathcal{U}_q(X_n)$ module with highest weight $2s\bar{\Lambda}_1$ and $\bar{\Lambda}_1$ is a fundamental weight of X_n.

In the next section we give two types of representation, ϱ and ϱ', of the TL algebra based on this formulation. The former is relevant to generic values of q while q is specialized to roots of unity in the latter. These representations give rise to several families of solvable quantum spin chains, vertex and IRF models which will be presented in section 3. The spin chain Hamiltonians (3.1a) correspond to the 'Hamiltonian limit' of the vertex models in section 3.2 and are $\mathcal{U}_q$-invariant, i.e. they commute with the quantum group action. Due to the underlying TL algebra, the vertex and IRF models are equivalent, in the sense of [2], with the 6-vertex and the 'Q-state' self-dual Potts models with Q given by (1.2)—as are the quantum spin chains with the spin-$\frac{1}{2}$ XXZ chain with appropriate coupling. However, the equivalence is at least at the level of the free or ground state energies, with each model possessing its own properties of individual interest to be investigated. In section 4 we provide one such analysis. In particular, we consider the IRF models corresponding to $Q > 4$ and follow the treatment of [4] to determine a correlation measuring the order of the system. We find that the calculation is reduced to the problem of diagonalizing a certain matrix (4.4) involving X_n characters. This is done in the appendix by exploiting a connection with the fusion rules in Wess–Zumino–Witten (WZW) conformal field theories [5]. The result (4.8) is neatly expressed in terms of the modular transformation matrix for $X_n^{(1)}$ characters.

We note that the quantum spin chains for the case $(V_{\bar{\xi}}, \mathcal{U}_q) = (V_{2s\bar{\Lambda}_1}, \mathcal{U}_q(A_1))$ have recently been studied by several authors. The limit $q \to 1$ has been discussed for general s [6, 7]. As for the related spin-1 biquadratic model [6,8], they are massive for $s \geq 1$ and of relevance to the dimerization transition on SU(n) antiferromagnetic chains [9]. The case $s = \frac{1}{2}$ has also been investigated in some detail in [10].

Our main aim in this paper is to elucidate specific features of interest arising from the framework of quantum groups. The resulting $\mathcal{U}_q$-invariance provides neat expressions for the eigenvectors of incidence matrices and one-point correlation functions for the IRF models given in (3.11) in terms of characters and fusion rules, etc. These IRF models belong to an already known class of so-called 'graph-state IRF' models for which one could do similar calculations in general but without such advantages.

2. Representations of the TL algebra and quantum groups

We begin by recalling some basic ingredients for the construction of representations of the TL algebra out of quantum groups. We describe two types of representation, ϱ and ϱ', which are related to vertex and IRF models, respectively. Unless otherwise stated we assume that q is generic, i.e. non-zero and not a root of unity and use the notation

$$[u] = \frac{q^u - q^{-u}}{q - q^{-1}}.$$

2.1. Representation ϱ related to vertex models

Let $\mathcal{U}_q = \mathcal{U}_q(X_n)$ be the quantized universal enveloping algebra of some finite dimensional classical Lie algebra X_n equipped with a co-product [11]

$$\Delta : \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q .$$

Further let $\pi : \mathcal{U}_q \to \mathrm{End}\, V_{\bar{\xi}}$ be its finite dimensional irreducible representation and assume that the decomposition of $V_{\bar{\xi}} \otimes V_{\bar{\xi}}$ is multiplicity free and includes one trivial representation on V_0. Denote by $\mathcal{P}_0 = \mathcal{P}_0^2$ the projector from $V_{\bar{\xi}} \otimes V_{\bar{\xi}}$ onto V_0. Then the following map ϱ is known to provide a representation of the TL algebra on $V_{\bar{\xi}}^{\otimes N}$ (cf [3])

$$\varrho : U_j \mapsto \sqrt{Q}(1 \otimes \cdots 1 \otimes \underbrace{\mathcal{P}_0}_{j\,,\,j+1} \otimes 1 \cdots \otimes 1) \qquad 1 \leq j < N \tag{2.1}$$

with Q as given in (1.2). By construction the $\varrho(U_j)$s belong to the commutant of $\mathcal{U}_q$, i.e. $[\varrho(U_j),\, \pi^{\otimes N}\Delta^{(N)}(\mathcal{U}_q)] = 0$, where $\Delta^{(N)} : \mathcal{U}_q \to \mathcal{U}_q^{\otimes N}$ is the co-product that naturally extends $\Delta = \Delta^{(2)}$.

From now on we restrict our consideration to the specific examples mentioned in the introduction, i.e. $(V_{\bar{\xi}}, \mathcal{U}_q(X_n)) = (V_{2s\bar{\Lambda}_1}, \mathcal{U}_q(A_1))$ with $2s \in \mathbf{Z}_{\geq 0}$ and $(V_{\bar{\Lambda}_1}, \mathcal{U}_q(B_n)), (V_{\bar{\Lambda}_1}, \mathcal{U}_q(C_n))$ and $(V_{\bar{\Lambda}_1}, \mathcal{U}_q(D_n))$. In these examples all of the weights in $V_{\bar{\xi}}$ are multiplicity free, which considerably simplifies the description given later. Our construction is based on the affinization $X_n^{(1)}$ of X_n as well as the q-deformation $\mathcal{U}_q(X_n)$. In the following, we shall always assume this correspondence among $X_n, \mathcal{U}_q(X_n)$ and $X_n^{(1)}$.

Let us fix some notations for the affine Lie algebras [12],

$$X_n^{(1)} = A_1^{(1)}, B_n^{(1)}, C_n^{(1)} \text{ and } D_n^{(1)}.$$

Let $\Lambda_j (0 \leq j \leq n)$ denote the fundamental weights and put $\mathcal{H}^* = \sum_{j=0}^{n} \mathbf{C}\Lambda_j$. We further set

$$\rho = \Lambda_0 + \cdots + \Lambda_n .$$

For any element $a \in \mathcal{H}^*$, we shall write $\bar{a}$ to signify its classical part. We will naturally label the finite-dimensional irreducible representations of $\mathcal{U}_q(X_n)$ by the elements in $\sum_{j=1}^{n} \mathbf{Z}_{\geq 0}\bar{\Lambda}_j$. We introduce orthonormal vectors ϵ_i, $(\epsilon_i, \epsilon_j) = \delta_{ij}$, and express the classical parts $\bar{\Lambda}_i$, $\bar{\rho}$ and the set $\mathcal{A}$ of weights appearing in the representation π of

$\mathcal{U}_q(X_n)$ as follows:

$$
\begin{aligned}
A_1: \quad & \mathcal{A} = \{s(\epsilon_1 - \epsilon_2), (s-1)(\epsilon_1 - \epsilon_2), \ldots, -s(\epsilon_1 - \epsilon_2)\} \\
& \bar{\Lambda}_1 = \tfrac{1}{2}(\epsilon_1 - \epsilon_2) \\
& \bar{\rho} = \tfrac{1}{2}(\epsilon_1 - \epsilon_2) \\
& J = \{s, s-1, \ldots, -s\} \\
B_n\,(n \geq 2): \quad & \mathcal{A} = \{0, \pm\epsilon_1, \ldots, \pm\epsilon_n\} \\
& \bar{\Lambda}_i = \epsilon_1 + \cdots + \epsilon_i \qquad (1 \leq i < n) \\
& \quad\; = \tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_n) \qquad (i = n) \\
& \bar{\rho} = (n - \tfrac{1}{2})\epsilon_1 + \cdots + \tfrac{1}{2}\epsilon_n \\
& J = \{0, \pm 1, \ldots, \pm n\} \\
C_n\,(n \geq 1): \quad & \mathcal{A} = \{\pm\epsilon_1, \ldots, \pm\epsilon_n\} \\
& \bar{\Lambda}_i = \epsilon_1 + \cdots + \epsilon_i \qquad (1 \leq i \leq n) \\
& \bar{\rho} = n\epsilon_1 + \cdots + \epsilon_n \\
& J = \{\pm 1, \ldots, \pm n\} \\
D_n\,(n \geq 3): \quad & \mathcal{A} = \{\pm\epsilon_1, \ldots, \pm\epsilon_n\} \\
& \bar{\Lambda}_i = \epsilon_1 + \cdots + \epsilon_i \qquad (1 \leq i < n-1) \\
& \quad\; = \tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n) \qquad (i = n-1) \\
& \quad\; = \tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n) \qquad (i = n) \\
& \bar{\rho} = (n-1)\epsilon_1 + \cdots + \epsilon_{n-1} \\
& J = \{\pm 1, \ldots, \pm n\}.
\end{aligned}
\tag{2.2}
$$

For $X_n = B_n, C_n$ and D_n, we extend the suffix of ϵ_μ to $-n \leq \mu \leq n$ by setting $\epsilon_{-\mu} = -\epsilon_\mu$ (hence $\epsilon_0 = 0$). We have also introduced the index set J in (2.2) so that $\mathcal{A} = \{\mu(\epsilon_1 - \epsilon_2) \mid \mu \in J\}$ for A_1 and $\mathcal{A} = \{\epsilon_\mu \mid \mu \in J\}$ for B_n, C_n and D_n. For each $\mu \in J$, let $v_\mu \in V_{\bar{\xi}}$ denote the normalized weight vector having the weight $\mu(\epsilon_1 - \epsilon_2)$ for A_1 and ϵ_μ for the other cases. Then the space V_0 that specifies the projector $\mathcal{P}_0$ in (2.1) is spanned by the following normalized vector

$$
Q^{-\frac{1}{4}} \sum_{\mu \in J} \varepsilon(\mu) q^{-<\epsilon_\mu, \bar{\rho}>} v_\mu \otimes v_{-\mu}
\tag{2.3}
$$

where $\varepsilon(\mu) = \pm 1$ is a sign factor depending on the choice of X_n as specified in table 1. We have also listed the values of $\sqrt{Q}$ along with related data for later use. In particular, ξ is the element of $\mathcal{H}^*$ whose classical part determines $V_{\bar{\xi}}$. Denoting the matrix unit by $E_{\mu\nu} \in \mathrm{End}\, V_{\bar{\xi}}$, $E_{\mu\nu} v_\kappa = \delta_{\nu\kappa} v_\mu$, the projector then takes the form

$$\mathcal{P}_0 = Q^{-1/2} \sum_{\mu,\nu \in J} \varepsilon(\mu)\varepsilon(\nu) q^{-<\epsilon_\mu+\epsilon_\nu,\bar{\rho}>} E_{\mu\nu} \otimes E_{-\mu-\nu}. \tag{2.4}$$

It follows from (2.1) that the TL relationships (1.1a) and (1.1c) are automatically satisfied among the $\varrho(U_j)$s. The ternary relationship (1.1b) may be verified by using the well-known diagramatical representation for $\varrho(U_j)$ as in figure 1.

Table 1. Data for the algebras under consideration. $\sqrt{Q}$ is the value in (1.2), t and g are the $|\text{long root}|^2/2$ and the dual Coxeter number, respectively. For $\mu \in J$ in (2.2), the symbol $\tilde{\mu}$ is defined by $\tilde{\mu} = \mu + (1 \pm 1)/4$ for A_1 with $s \in \mathbf{Z} + (1 \pm 1)/4$ and $\tilde{\mu} = 0$ with the exception $\tilde{0} = 1$ for B_n

	A_1	B_n	C_n	D_n
$\sqrt{Q}$	$[2s+1]$	$[2n-1][n+\frac{1}{2}]/[n-\frac{1}{2}]$	$[n][2n+2]/[n+1]$	$[2n-2][n]/[n-1]$
ξ	$(l-2s)\Lambda_0 + 2s\Lambda_1$	$(l-1)\Lambda_0 + \Lambda_1$	$(l-1)\Lambda_0 + \Lambda_1$	$(l-1)\Lambda_0 + \Lambda_1$
$\varepsilon(\mu)$	$(-)^{\tilde{\mu}}$	$(-)^{\tilde{\mu}}$	$\mathrm{sgn}\,\mu$	1
t	1	1	2	1
g	2	$2n-1$	$n+1$	$2n-2$
$h(a)$		$[a]$	$[2a]$	1
ζ	$1/\sqrt{2L}$	$1/2\sqrt{L^n}$	$1/\sqrt{L^n}$	$1/2\sqrt{L^n}$

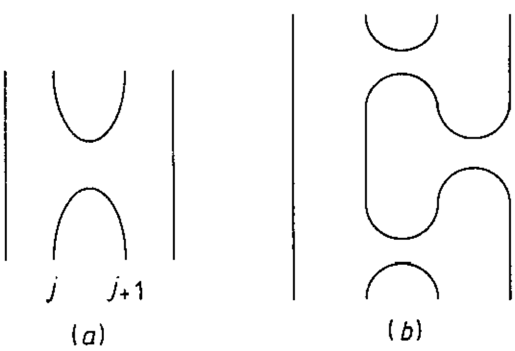

Figure 1. (a) Diagramatical representation of $\rho(U_j)$. (b) Representation of the left-hand side of the ternary relationship (1.1b). The right-hand side is recovered by straightening the string.

We note that there are in fact multi-parameter representations of the TL algebra. They are obtained by replacing (2.4) with

$$\sum_{\mu,\nu \in J} z_\mu z_\nu E_{\mu\nu} \otimes E_{-\mu-\nu} \tag{2.5}$$

where the parameters z_μ obey the constraints

$$z_\mu z_{-\mu} = \frac{1}{\sqrt{Q}} \qquad \sum_{\mu \in J} z_\mu^2 = 1. \tag{2.6}$$

Regarding Q as a dependent variable of the z_μs, the number of free parameters for $\mathcal{U}_q(A_1)$ is s (respectively $s+\frac{1}{2}$) when $2s$ is even (respectively odd) and is equal to the rank n in the remaining cases. Using this representation one can also construct solvable quantum spin chains and vertex models by the prescriptions given in (3.1a) and (3.5a). However, in that case the $\mathcal{U}_q$-invariance, (3.1b) and (3.7), no longer hold. The existence of a multi-parameter representation of the TL algebra for the case $\mathcal{U}_q(A_1)$ has recently been noted independently in [13].

2.2. Representation ϱ' related to IRF models

Let us now proceed to another representation ϱ' of the TL algebra. This time, the basis of the representation space is labelled by a 'path' of the dominant integral weights (DIWs) of $X_n^{(1)}$ with a fixed level. We let $l \in \mathbf{Z}_{\geq 0}$ denote the level and introduce the quantity $L = t(l+g)$, where t and g are the |long root|$^2/2$ and the dual Coxeter number listed in table 1. Then the DIWs have the following forms:

$$\begin{aligned}
A_1^{(1)}:&\quad a = (L - a_1 - 1)\Lambda_0 + (a_1 - 1)\Lambda_1 \\
&\quad a_1 \in \mathbf{Z},\ 0 < a_1 < L \\
B_n^{(1)}:&\quad a = (L - a_1 - a_2 - 1)\Lambda_0 + \sum_{i=1}^{n-1}(a_i - a_{i+1} - 1)\Lambda_i + (2a_n - 1)\Lambda_n \\
&\quad \forall a_i \in \mathbf{Z} \text{ or } \forall a_i \in \mathbf{Z} + \tfrac{1}{2},\ L > a_1 + a_2,\ a_1 > a_2 > \cdots > a_n > 0 \\
C_n^{(1)}:&\quad a = (L/2 - a_1 - 1)\Lambda_0 + \sum_{i=1}^{n-1}(a_i - a_{i+1} - 1)\Lambda_i + (a_n - 1)\Lambda_n \\
&\quad \forall a_i \in \mathbf{Z},\ L/2 > a_1 > a_2 > \cdots > a_n > 0 \\
D_n^{(1)}:&\quad a = (L - a_1 - a_2 - 1)\Lambda_0 + \sum_{i=1}^{n-1}(a_i - a_{i+1} - 1)\Lambda_i + (a_{n-1} + a_n - 1)\Lambda_n \\
&\quad \forall a_i \in \mathbf{Z} \text{ or } \forall a_i \in \mathbf{Z} + \tfrac{1}{2},\ L > a_1 + a_2,\ a_1 > a_2 > \cdots > a_n,\ a_{n-1} + a_n > 0.
\end{aligned} \tag{2.7}$$

Given l, let $P_+(l)$ denote the set of level l DIWs whose elements are as specified previously. We set $\bar{P}_+ = \cup_{l\geq 0}\overline{P_+(l)} = \sum_{i=1}^{n}\mathbf{Z}_{\geq 0}\bar{\Lambda}_i$. From (2.2) and (2.7) the classical part $\bar{a}$ can be expressed as $\bar{a} + \bar{\rho} = \frac{1}{2}a_1(\epsilon_1 - \epsilon_2)$ for $A_1^{(1)}$ and $\bar{a} + \bar{\rho} = a_1\epsilon_1 + \cdots + a_n\epsilon_n$ for the other cases. For $\bar{a} \in \bar{P}_+$, let $V_{\bar{a}}$ denote the irreducible $\mathcal{U}_q$ module with highest weight $\bar{a}$. We define its q-dimension by

$$\chi_a = \mathrm{Tr}_{V_{\bar{a}}}(q^{-2\bar{\rho}}). \tag{2.8}$$

Explicitly, it is given by

$$\chi_a = \frac{A_{\bar{a}+\bar{\rho}}}{A_{\bar{\rho}}} \tag{2.9a}$$

$$A_{\bar{a}+\bar{\rho}} = \begin{cases} [a_1] & \text{for } A_1 \quad (2.9b) \\ \prod_{1\leq i\leq n} h(a_i) \prod_{1\leq i<j\leq n} [a_i + a_j][a_i - a_j] & \text{for } B_n, C_n \text{ and } D_n \quad (2.9c) \end{cases}$$

where the function $h(a)$ is available in table 1. These are the q-analogues of Weyl's dimension formulae. For a generic element $\bar{a} \in \bar{P}_+$, the irreducible decomposition of $V_{\bar{a}} \otimes V_{\bar{\xi}}$ has the form

$$V_{\bar{a}} \otimes V_{\bar{\xi}} = \oplus_{\bar{b} \in \bar{P}_+, \bar{b}-\bar{a} \in \mathcal{A}} V_{\bar{b}}. \tag{2.10}$$

(For $\mathcal{U}_q(B_n)$ with $a_n = \frac{1}{2}$, the term corresponding to $\bar{b} = \bar{a}$ on the right-hand side does not appear.) By evaluating $q^{-2\bar{\rho}}$ on both sides of (2.10), we get a character identity

$$\sqrt{Q}\chi_a = \sum_{\bar{b} \in \bar{P}_+, \bar{b}-\bar{a} \in \mathcal{A}} \chi_b. \tag{2.11}$$

Now we specialize the deformation parameter q to a root of unity as follows

$$q = e^{\pi i/L} \qquad L = t(l+g). \tag{2.12}$$

We assume that $L \geq 2s+3$ for $\mathcal{U}_q(A_1)$. Given the elements $a, b \in \mathcal{H}^*$, we shall call the pair (a, b) *admissible* if and only if the following conditions hold. (The order of a and b is actually irrelevant in the present cases because $\mathcal{A} = -\mathcal{A}$ as a set.)

$$\begin{aligned}
A_1^{(1)}: \quad & a, b \in P_+(l) \\
& a_1 - b_1 \in \{-2s, -2s+2, \ldots, 2s\} \\
& a_1 + b_1 \in \{2s+2, 2s+4, \ldots, 2L-2s-2\} \\
& \text{where } \bar{a} = (a_1-1)\bar{\Lambda}_1,\ \bar{a} = (b_1-1)\bar{\Lambda}_1 \\
B_n^{(1)}: \quad & a, b \in P_+(l),\ b - a \in \mathcal{A} \\
& b \neq a \text{ if } a_n = \tfrac{1}{2} \\
C_n^{(1)}, D_n^{(1)}: \quad & a, b \in P_+(l),\ b - a \in \mathcal{A}.
\end{aligned} \tag{2.13}$$

These conditions have appeared in the solvable RSOS models in [14] and are also known as the fusion rules in WZW conformal field theories (see the appendix). We will utilize this fact in section 4 in evaluating some physical quantities of the relevant IRF models. The admissibility conditions (2.13) are neatly described by incidence diagrams, as depicted in figure 2.

Suppose that $a \in P_+(l)$. Then under the specialization (2.12) one can show that $\sum_b^* \chi_b = 0$, where the sum is taken over the level l integral weights b (not necessarily DIWs) of $X_n^{(1)}$ such that $b - a \in \mathcal{A}$ and (a, b) is not admissible. Combining this with (2.11) one can show

$$\sqrt{Q}\chi_a = \sum_{b \in P_+(l),\, (a,b):\text{admissible}} \chi_b \qquad \text{for } \forall a \in P_+(l). \tag{2.14}$$

Consider now a $|P_+(l)|$-dimensional vector space $\mathcal{E} = \sum_{a \in P_+(l)} \mathbb{C} e_a$ whose base vectors are labelled by the DIWs. Our representation space Ψ of the TL algebra is defined to be the subspace $\mathcal{E}^{\otimes N+1}$ spanned by the vectors $e_{a(0)} \otimes e_{a(1)} \otimes \cdots \otimes e_{a(N)}$ such that

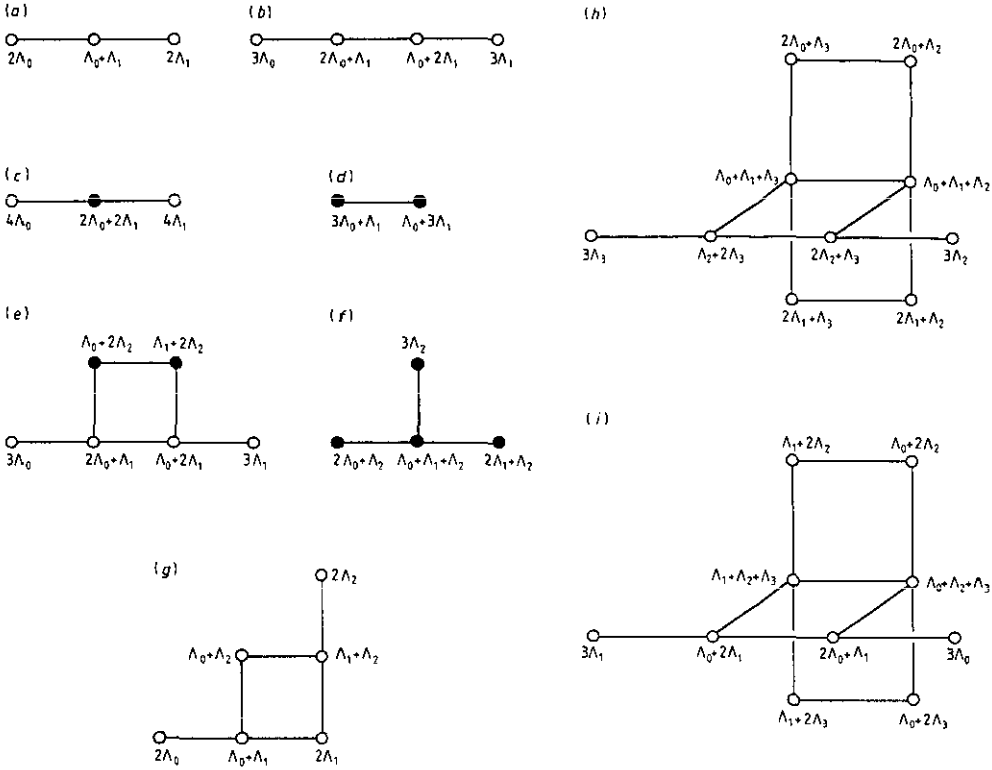

Figure 2. Incidence diagrams. Each node (either filled or empty) corresponds to a DIW. A bond connects two nodes if and only if they are admissible. A filled node is admissible to itself. The examples shown are: (a) $A_1^{(1)}$ level 2, $s = \frac{1}{2}$ (Ising); (b) $A_1^{(1)}$ level 3, $s = \frac{1}{2}$ (hard hexagon); (c) and (d) $A_1^{(1)}$ level 4, $s = 1$ (split into two sectors); (e) and (f) $B_2^{(1)}$ level 3, (split into two sectors according to whether $a_i \in \mathbf{Z}$ or $\mathbf{Z} + \frac{1}{2}$), (g) $C_2^{(1)}$ level 2, (h) and (i) $D_3^{(1)}$ level 3, (split into two sectors as in $B_2^{(1)}$).

$(a^{(j)}, a^{(j+1)})$ is admissible for $0 \le j < N$. Then the matrices $\varrho'(U_j) \in \mathrm{End}\,\Psi$, $(1 \le j < N)$ defined by the following realize the TL relationships (1.1).

$$\begin{aligned}\varrho'(U_j) e_{a^{(0)}} \otimes \cdots \otimes e_{a^{(j-1)}} \otimes e_{a^{(j)}} \otimes e_{a^{(j+1)}} \otimes \cdots \otimes e_{a^{(N)}} \\ = \delta_{a^{(j-1)}\, a^{(j+1)}} \sum_d \sqrt{\frac{\chi_{a^{(j)}} \chi_d}{\chi_{a^{(j-1)}} \chi_{a^{(j+1)}}}} \\ \times e_{a^{(0)}} \otimes \cdots \otimes e_{a^{(j-1)}} \otimes e_d \otimes e_{a^{(j+1)}} \otimes \cdots \otimes e_{a^{(N)}}.\end{aligned} \tag{2.15}$$

Here the summation on the right-hand side is taken over $d \in P_+(l)$ under the condition that $(d, a^{(j\pm1)})$ is admissible. The ternary relationship (1.1b) as well as (1.1c) follow straightforwardly and (1.1a) is reduced to the identity (2.14). We note that $\chi_a > 0$ for $a \in P_+(l)$. The representation ϱ' is related to ϱ in section 2.1. The former is effectively derivable from the latter by a certain base change based on the q-analogue of Brauer–Weyl reciprocity (cf [3]).

3. Quantum spin chains, vertex and IRF models

3.1. Quantum spin chains

Consider a one-dimensional lattice populated with an interacting 'spin' at each site $1 \leq j \leq N$. Specifically, the spin variables range over the set of weight vectors $\{v_\mu \mid \mu \in J\}$ and thus our Hilbert space is an N-fold tensor product $V_{\bar\xi} \otimes \cdots \otimes V_{\bar\xi}$. For $\mathcal{U}_q = \mathcal{U}_q(\mathrm{sl}(2))$, these are the q-analogues of the usual SU(2) spin states. The Hamiltonians associated with the representation ϱ in (2.1) are sums of the TL operators:

$$H = \text{constant} \sum_{j=1}^{N-1} \varrho(U_j) \tag{3.1a}$$

where for convenience we have assumed free boundary conditions. By definition it commutes with the $\mathcal{U}_q$-action:

$$[H, \pi^{\otimes N} \Delta^{(N)}(\mathcal{U}_q)] = 0. \tag{3.1b}$$

This Hamiltonian has an alternative expression in terms of Casimir operators, which we shall now explain. Let $C \in \mathcal{U}_q$ be a Casimir element $[C, \mathcal{U}_q] = 0$ and put $X = \Delta(C) - 1 \otimes C - C \otimes 1 \in \mathcal{U}_q \otimes \mathcal{U}_q$. We shall write $C(\bar\Lambda)$ to signify its value $C|_{V_{\bar\Lambda}}$ on the irreducible $\mathcal{U}_q$ module with highest weight $\bar\Lambda$. The decomposition of $V_{\bar\xi} \otimes V_{\bar\xi}$ under consideration has the form

$$\begin{aligned} V_{\bar\xi} \otimes V_{\bar\xi} &= \oplus_{\bar\Lambda \in \mathcal{D}} V_{\bar\Lambda} \\ \mathcal{D} &= \{4s\bar\Lambda_1, (4s-2)\bar\Lambda_1, \ldots, 2\bar\Lambda_1, 0\} \qquad \text{for } \mathcal{U}_q(A_1) \\ &= \{2\bar\Lambda_1, \bar\Lambda_2, 0\} \qquad \text{for } \mathcal{U}_q(B_n), \mathcal{U}_q(C_n), \mathcal{U}_q(D_n). \end{aligned} \tag{3.2}$$

For each $\bar\Lambda \in \mathcal{D}$, we write $X(\bar\Lambda)$ to specify the value of X on $V_{\bar\Lambda} \subset V_{\bar\xi} \otimes V_{\bar\xi}$; thus we have $X(\bar\Lambda) = C(\bar\Lambda) - 2C(\bar\xi)$. Define a polynomial $f(x)$ and a matrix $X_j \in \mathrm{End}\,(V_{\bar\xi}^{\otimes N})$, $(1 \leq j < N)$ by

$$f(x) = \sqrt{Q} \prod_{\bar\Lambda \in \mathcal{D},\, \bar\Lambda \neq 0} \left(\frac{x - X(\bar\Lambda)}{X(0) - X(\bar\Lambda)} \right) \tag{3.3a}$$

$$X_j = 1 \otimes \cdots 1 \otimes \underbrace{(\pi \otimes \pi) f(X)}_{j,\,j+1} \otimes 1 \cdots \otimes 1. \tag{3.3b}$$

Assuming the non-degeneracy in the spectrum of $X(\bar\Lambda)$, $X_j/\sqrt{Q}$ is by definition the projector $\mathcal{P}_0$ itself acting on the $j, (j+1)$th slot, therefore from (2.1) we have

$$X_j = \varrho(U_j) \tag{3.4}$$

which yields an alternative expression for the Hamiltonian (3.1a) in terms of Casimir operators.

A few remarks are in order for the case $\mathcal{U}_q(A_1)$. When $q = 1$ and $s = \frac{1}{2}$ the operator X_j is essentially the Heisenberg interaction term $\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z$. Besides the explicit matrix elements, available from (2.1), (2.2) and (2.4), the local Hamiltonian

$X_j = \varrho(U_j)$ is in principle also expressible in terms of the usual representation matrices for SU(2) generators starting from the general formula (3.3). The resulting q-deformed Hamiltonian has been written down for $s = 1$ in [15], see also [16]. However for $s > 1$, the representation matrices for $\mathcal{U}_q(A_1)$ are no longer proportional to their classical counterparts ($q = 1$) and writing the Hamiltonian in terms of the usual SU(2) operators becomes very cumbersome. The formula (3.3) and the corresponding value of Q, $\sqrt{Q} = [2s+1]$, provide the anisotropic generalization of the results obtained in [7] for $q = 1$. However, the SU(n) Hamiltonians discussed in [9] do not belong to the present family for $n \geq 3$ since in that case the Hilbert space is the alternating tensor product $V \otimes V' \otimes \cdots V \otimes V'$, where V' has the conjugated highest weight to that of V.

The Hamiltonians for the remaining cases $\mathcal{U}_q(B_n)$, $\mathcal{U}_q(C_n)$ and $\mathcal{U}_q(D_n)$ appear to be new although due to the TL equivalence, they are expected to possess the same thermodynamic properties as the spin-$\frac{1}{2}$ XXZ chain with appropriate coupling.

3.2. Vertex models

Using the representation ϱ (2.1) of the TL algebra, one can also build solvable vertex models whose Hamiltonian limit leads to the previously mentioned quantum spin chains. To do so, we introduce an operator $R(u) \in \mathrm{End}\,(V_{\bar{\xi}} \otimes V_{\bar{\xi}})$ by

$$R(u) = \frac{\sinh(\eta - u)}{\sinh \eta} id_{V_\xi \otimes V_\xi} + \sqrt{Q}\frac{\sinh u}{\sinh \eta}\mathcal{P}_0 = \sum R(u,q)_{\mu\nu\kappa\lambda} E_{\mu\nu} \otimes E_{\kappa\lambda} \tag{3.5a}$$

$$R(u,q)_{\mu\nu\kappa\lambda} = \frac{\sinh(\eta - u)}{\sinh \eta}\delta_{\mu\nu}\delta_{\kappa\lambda} + \frac{\sinh u}{\sinh \eta}\varepsilon(\mu)\varepsilon(\nu)q^{-<\epsilon_\mu + \epsilon_\nu, \bar{\rho}>}\delta_{\mu\,-\kappa}\delta_{\nu\,-\lambda} \tag{3.5b}$$

where $u \in \mathbf{C}$ is the spectral parameter and the 'anisotropy parameter' η is chosen so that

$$2\cosh \eta = \sqrt{Q}. \tag{3.6}$$

The matrix $R(u)$ commutes with the quantum group action, i.e.

$$[R(u), \pi^{\otimes 2}\Delta(\mathcal{U}_q)] = 0. \tag{3.7}$$

Setting

$$\begin{aligned} R_j(u) &= 1 \otimes \cdots 1 \otimes \underbrace{R(u)}_{j,j+1} \otimes 1 \cdots \otimes 1 \\ &= \frac{\sinh(\eta - u)}{\sinh \eta} id_{V_\xi^{\otimes N}} + \frac{\sinh u}{\sinh \eta}\varrho(U_j) \qquad 1 \leq j < N \end{aligned} \tag{3.8}$$

one can readily show that the Yang–Baxter equation

$$R_{j+1}(u)R_j(u+v)R_{j+1}(v) = R_j(v)R_{j+1}(u+v)R_j(u) \tag{3.9}$$

is valid owing to the TL relations among the $\varrho(U_j)$s. Thus the matrix $R(u)$ gives rise to a solvable vertex model on a planar square lattice with the following features:

(1) the local states range over the set $\{v_\mu\}_{\mu \in J}$ of the base vectors of $V_{\bar{\xi}}$; and

(2) the Boltzmann weight of the configuration $v_\nu, v_\lambda, v_\kappa, v_\mu$ round a vertex (ordered clockwise from the upper bond) is given by $R(u,q)_{\mu\nu\kappa\lambda}$ in (3.5*b*).

As a result of (1) the number of the states is given by $\dim V_{\bar{\xi}}(= \lim_{q\to 1}\sqrt{Q}) = 2s+1$ for $A_1^{(1)}$, $2n+1$ for $B_n^{(1)}$, $2n$ for $C_n^{(1)}$ and $D_n^{(1)}$. Due to the TL algebra the model is equivalent to the 6-vertex and the 'Q-state' self-dual Potts models through the argument in [2]. In fact, the cases $\mathcal{U}_q(A_1)$ with $s = \frac{1}{2}$ and $\mathcal{U}_q(C_1)$ yield the 6-vertex model itself. When $q = 1$, the vertex models here reduce to those discussed in [17] (see also [6]), where the number of the states is equal to the square root of that for the equivalent Potts model. The Hamiltonians (3.1*a*) are readily recovered by taking the logarithmic derivative of the row-to-row transfer matrix with respect to u.

3.3. IRF models

Let us turn to the IRF models related to the representation ϱ' (2.15). Define an operator $W_j(u) \in \mathrm{End}\,\Psi$, $(1 \le j < N)$ by

$$W_j(u) = \frac{\sinh(\eta - u)}{\sinh \eta} id_\Psi + \frac{\sinh u}{\sinh \eta}\varrho'(U_j) \tag{3.10}$$

where the parameter η is chosen as in (3.6) with the specialization (2.12) on the right-hand side. By virtue of the TL relations among the $\varrho'(U_j)$s, the Yang–Baxter equation (3.9) is valid again for the $W_j(u)$s. As a result we obtain solvable IRF models with the following features:

(1) the local states range over the set $P_+(l)$ (2.7) of the DIWs of $X_n^{(1)}$;

(2) the adjacent pairs of local states (a, b) must be admissible as specified in (2.13); and

(3) the Boltzmann weight $W_u\begin{pmatrix} a & b \\ d & c\end{pmatrix}$ of the face configuration (a at the NW corner of the face, etc) is given by

$$W_u\begin{pmatrix} a & b \\ d & c\end{pmatrix} = \begin{cases} \dfrac{\sinh(\eta-u)}{\sinh\eta}\delta_{bd} + \dfrac{\sinh u}{\sinh \eta}\delta_{ac}\sqrt{\dfrac{\chi_b\chi_d}{\chi_a\chi_c}} \\ \qquad \text{if } (a,b),(b,c),(a,d) \text{ and } (d,c) \text{ are admissible} \\ 0 \qquad \text{otherwise} \end{cases} \tag{3.11}$$

where the χ_as are those in (2.9) under the specialization (2.12).

Conditions (1) and (2) are depicted in the incidence diagrams of figure 2. Each node therein corresponds to a local state. Two states can occupy adjacent lattice sites if and only if the corresponding nodes are connected by a bond.

Using (3.7), one can relate the present IRF models to the vertex models in section 3.2 through the 'vertex–IRF correspondence' in the sense of [18]. For $\mathcal{U}_q(A_1)$ with $s = \frac{1}{2}$ and $\mathcal{U}_q(C_1)$ the present construction gives the trigonometric limit of the 8-vertex solid-on-solid models [19]. The corresponding values of Q are given by the Beraha numbers, $Q = 4\cos^2\pi/L$, that appeared in the work [20] as observed in [21, 22]. Models having the structure (3.11) have also been discussed in [4, 22, 23]. Due to the TL equivalence with the 'Q-state' self-dual Potts model, they are critical for $Q \le 4$ and at a first-order transition for $Q > 4$.

4. IRF models with $Q > 4$

In this section we investigate the order in the IRF models for $Q > 4$. From table 1 and (2.12) we find that the following cases correspond to such a situation,

$$\begin{aligned} A_1^{(1)}: &\quad s = 1, l \geq 5, \text{ or } s \geq \tfrac{3}{2}, l \geq 2s+2 \\ B_n^{(1)}: &\quad n \geq 2, l \geq 3 \\ C_n^{(1)}: &\quad n \geq 2, l \geq 2 \\ D_n^{(1)}: &\quad n \geq 3, l \geq 3. \end{aligned} \tag{4.1}$$

As previously mentioned, these models should be at a first-order transition point because of their TL equivalence with the Potts model. This implies the existence of a broken symmetry or order within the system. Let us evaluate the correlation between a boundary state $a^{(\infty)}$ and a state $a^{(0)}$ deep within the lattice $\mathcal{L}$ through the method of [4]. In the following, we let $\sqrt{Q} > 2$ be as in table 1 under the specialization (2.12), fix the parameter η by (3.6) with $\eta > 0$, and assume that $0 < u < \eta$.

Consider the partition function of the model

$$Z = \sum_{\text{configurations}} \prod_{\text{faces}} W_u \begin{pmatrix} a^{(i)} & a^{(j)} \\ a^{(l)} & a^{(k)} \end{pmatrix} \tag{4.2}$$

where the product extends over all faces of the lattice $\mathcal{L}$. Substituting (3.11) into this one gets an expansion of Z in terms of Kronecker deltas. Placing a diagonal bond connecting the sites i and k (j and l) when picking up terms involving $\delta_{a^{(i)}\,a^{(k)}}$ ($\delta_{a^{(j)}\,a^{(l)}}$), each summand in the expansion is specified by a bond graph on $\mathcal{L}$. The lattice sites connected by a bond therein take the same state, forming a cluster. We fix the boundary effects so that every bond graph consists of an infinite boundary cluster and some finite clusters contained in it. Let $a^{(\infty)}$ be a site variable belonging to the boundary cluster and $a^{(0)}$ be the one deep within the lattice $\mathcal{L}$. We define the correlation $\mathcal{F}_{ab}$ by

$$\mathcal{F}_{ab} = Z^{-1} \sum_{\text{configurations}} \delta_{a a^{(0)}} \delta_{b a^{(\infty)}} \prod_{\text{faces}} W_u \begin{pmatrix} a^{(i)} & a^{(j)} \\ a^{(l)} & a^{(k)} \end{pmatrix} \tag{4.3}$$

for any $a, b \in P_+(l)$. It is the probability of finding the configurations such that $a^{(0)} = a$ under the condition that the boundary cluster assumes the value b. It is well known that the bond graphs on $\mathcal{L}$ are in one-to-one correspondence with the so-called polygon decompositions on its dual lattice (cf [24]). In [4], Owczarek and Baxter have extensively used this fact and obtained a formula expressing the correlation of this kind in terms of 'generalized percolation probabilities'. It applies quite generally to the TL type IRF models having the structure (3.11) with $Q > 4$. To adapt it to the present models, we define a $|P_+(l)| \times |P_+(l)|$ matrix $\mathcal{V}$ by

$$\begin{aligned} \mathcal{V} &= (\mathcal{V}_{ab})_{a,b \in P_+(l)} \\ \mathcal{V}_{ab} &= \frac{\chi_b}{\chi_a} \qquad \text{if } (a,b) \text{ is admissible} \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{4.4}$$

We also define $Y = (Y_{ab})_{a,b\in P_+(l)}$ to be the matrix that diagonalizes $\mathcal{V}$ and denote by $\{\lambda_c \mid c \in P_+(l)\}$ the complete set of eigenvalues of $\mathcal{V}$. Then

$$Y^{-1}\mathcal{V}Y = \mathrm{diag}(\lambda_c)_{c\in P_+(l)}. \tag{4.5}$$

We regard the lattice $\mathcal{L}$ as consisting of two sublattices in the usual way and set $r = 0$ or 1 according to whether $a^{(0)}$ and $a^{(\infty)}$ belong to the same sublattice or not. Under this setting, we apply (6.7) in [4] (with minor modification) and deduce the following expression for the correlation $\mathcal{F}_{ab}$ (4.3):

$$\mathcal{F}_{ab} = \sum_{c\in P_+(l)} Y_{bc}(Y^{-1})_{ca}G_r(\lambda_c) \tag{4.6a}$$

$$G_0(z) = \prod_{j=1}^{\infty}\left(\frac{1+\mathrm{e}^{-(8j-4)\eta}(z^2-2)+\mathrm{e}^{-(16j-8)\eta}}{1+\mathrm{e}^{-(8j-4)\eta}(Q-2)+\mathrm{e}^{-(16j-8)\eta}}\right) \tag{4.6b}$$

$$G_1(z) = \frac{z}{\sqrt{Q}}\prod_{j=1}^{\infty}\left(\frac{1+\mathrm{e}^{-8j\eta}(z^2-2)+\mathrm{e}^{-16j\eta}}{1+\mathrm{e}^{-8j\eta}(Q-2)+\mathrm{e}^{-16j\eta}}\right). \tag{4.6c}$$

The quantities $G_0(z)$ and $G_1(z)$ are first obtained in [25] as the generating functions for 'generalized percolation probabilities' in the dichromatic formulation of the self-dual Potts model. Mathematically, they are ratios of the level 1 $A_1^{(1)}$ characters (cf [12] p 217)

$$G_r(y^{1/2}+y^{-1/2}) = \frac{\cosh^{A_1^{(1)}}_{\Lambda_r}(y,\mathrm{e}^{-4\eta})}{\cosh^{A_1^{(1)}}_{\Lambda_r}(\mathrm{e}^{-2\eta},\mathrm{e}^{-4\eta})} \qquad r=0,1 \tag{4.7a}$$

$$\cosh^{A_1^{(1)}}_{\Lambda_r}(y,p) = \frac{\sum_{k\in\mathbf{Z}+\frac{1}{2}r} p^{k^2}y^{-k}}{\prod_{k=1}^{\infty}(1-p^k)} \tag{4.7b}$$

where the subscript Λ_r ($r = 0, 1$) signifies the highest weight of the level 1 $A_1^{(1)}$ modules. In this way the problem is reduced to finding the matrix Y in (4.5) that diagonalizes the matrix $\mathcal{V}$ in (4.4). This has been done in the appendix: see (A9). In particular, we have exploited the fact that the admissibility condition in (2.13) is nothing but the *fusion rule* in level l $X_n^{(1)}$ WZW model in conformal field theory [5]. With the use of Verlinde's formula [26] we arrive at the result

$$\mathcal{F}_{ab} = \frac{S_{a\,l\Lambda_0}}{S_{b\,l\Lambda_0}}\sum_{c\in P_+(l)} S_{bc}S^*_{ac}G_r\left(\frac{S_{\xi\,c}}{S_{l\Lambda_0\,c}}\right) \tag{4.8a}$$

where $S = (S_{ab})_{a,b\in P_+(l)}$ is given in (A3). It is the symmetric unitary matrix describing a modular transformation rule of the level l $X_n^{(1)}$ characters. In view of (4.5), (4.6a) and (A8), this result can also be expressed as the matrix element

$$\mathcal{F}_{ab} = G_r(\mathcal{V})_{ba} = \chi_a\chi_b^{-1}G_r(N_\xi)_{ba} \tag{4.8b}$$

where N_ξ is the fusion rule matrix (A7b). The correct normalization $\sum_a \mathcal{F}_{ab} = 1$ has been achieved due to the identity

$$\chi_b = \sum_{a\in P_+(l)} G_r(N_\xi)_{ba}\chi_a \tag{4.9}$$

which immediately follows from $G_r(\sqrt{Q}) = 1$ and (2.14), i.e.

$$\sqrt{Q}\chi_b = \sum_{a\in P_+(l)} (N_\xi)_{ba}\chi_a. \qquad (4.10)$$

The result (4.8) gives a description of the order in the IRF models for the cases listed in (4.1).

Acknowledgments

One of the authors (MTB) thanks Professor N Y Reshetikhin and Dr A Klümper for a useful discussion. This work has been supported by the Australian Research Council through the award of Queen Elizabeth II Research Fellowships.

Appendix. Diagonalization of the matrix $\mathcal{V}$

Here we diagonalize the matrix $\mathcal{V}$ defined in (4.4) and thereby determine Y and λ_c in (4.5). Clearly one has to convert the condition 'admissible' in (4.4) into a more manageable form. The key to doing this is the observation that the admissibility (2.13) is nothing other than the fusion rule in the level l $X_n^{(1)}$ WZW model in conformal field theory [5]. More precisely, we have the quantity N_{ac}^b determined from any three DIWs $a, b, c \in P_+(l)$ such that

$$\begin{aligned} N_{a\xi}^{b} &= 1 \qquad \text{if } (a,b) \text{ is admissible} \\ &= 0 \qquad \text{otherwise.} \end{aligned} \qquad (A1)$$

Here $\xi \in P_+(l)$ has been listed in table 1. It is the DIW whose classical part $\bar{\xi}$ specifies the basic constituent $V_{\bar{\xi}}$ of our formulation in the main text. Furthermore, the $N_{a\xi}^b$ has the following expression known as Verlinde's formula [26]:

$$N_{a\xi}^{b} = N_{\xi a}^{b} = \sum_{d\in P_+(l)} \frac{S_{\xi d} S_{a d} S_{b d}^{*}}{S_{l\Lambda_0 d}} \qquad (A2)$$

where $*$ stands for the complex conjugate and $S = (S_{ab})_{a,b\in P_+(l)}$ is the modular transformation ($\tau \to -1/\tau$) matrix of the level l $X_n^{(1)}$ characters. The elements are essentially specialized characters of the classical part X_n evaluated at $\bar{a} + \bar{\rho}$ [12].

$$S_{ab} = \zeta \prod_{\alpha\in\bar{\Delta}_+} \left(2 \sin\frac{\pi}{L} < \bar{a} + \bar{\rho}, \alpha >\right) \ \mathrm{Tr}_{V_{\bar{b}}}\left(\mathrm{e}^{-2\pi\mathrm{i}(\bar{a}+\bar{\rho})/L}\right) \qquad (A3)$$

where $\bar{\Delta}_+$ denotes the set of positive roots of X_n and ζ is a constant specified in table 1 and independent of a and b. For example in $A_1^{(1)}$, one has

$$S_{ab} = \sqrt{\frac{2}{l+2}} \sin\frac{\pi a_1 b_1}{l+2}$$

for $a = (l+1-a_1)\Lambda_0 + (a_1-1)\Lambda_1$ and $b = (l+1-b_1)\Lambda_0 + (b_1-1)\Lambda_1$. For the other algebras, explict formulae in terms of determinants can be found in [27]. The S is a symmetric unitary matrix, i.e.

$$S_{ab} = S_{ba} \qquad \sum_{c \in P_+(l)} S^*_{ac} S_{bc} = \delta_{ab}. \tag{A4}$$

In particular, $S_{l\Lambda_0\, a}$ is known to be real and positive for any $a \in P_+(l)$. From (A2) and (A4) we have

$$\sum_{b \in P_+(l)} N^b_{a\,\xi} S_{bc} = S_{ac} \frac{S_{\xi c}}{S_{l\Lambda_0\, c}}. \tag{A5}$$

Comparing (A3) with (1.2) and (2.8) under the specialization (2.12), we also get

$$\sqrt{Q} = \frac{S_{l\Lambda_0\, \xi}}{S_{l\Lambda_0\, l\Lambda_0}} \tag{A6a}$$

$$\chi_a = \frac{S_{l\Lambda_0\, a}}{S_{l\Lambda_0\, l\Lambda_0}}. \tag{A6b}$$

In terms of $|P_+(l)| \times |P_+(l)|$ matrices N_ξ and χ defined by

$$\chi = \ \mathrm{diag}\, (\chi_a)_{a \in P_+(l)} \tag{A7a}$$

$$N_\xi = \left(N^b_{a\,\xi}\right)_{a,b \in P_+(l)} \tag{A7b}$$

the matrix $\mathcal{V}$ in (4.4) is written as

$$\mathcal{V} = \chi^{-1} N_\xi\, \chi. \tag{A8}$$

Therefore from (4.5), the λ_cs are in fact eigenvalues of the fusion rule matrix N_ξ and χY is the one diagonalizing it. Thus from (A5) we find the solution

$$Y_{bc} = \frac{S_{bc}}{S_{l\Lambda_0\, b}} \qquad (Y^{-1})_{ca} = S^*_{ac} S_{l\Lambda_0\, a} \qquad \lambda_c = \frac{S_{\xi c}}{S_{l\Lambda_0\, c}}. \tag{A9}$$

This completes the diagonalization of the matrix $\mathcal{V}$.

Note added in proof. The authors thank Dr T Deguchi for informing us that the multi-parameter representations (2.5) and (2.6) have also been obtained in [28]. Very recently, $\mathcal{U}_q(\mathrm{SU}(2))$-invariant Hamiltonians were discussed in [29] and written in terms of classical spin generators for spin $\frac{3}{2}$. The IRF models of structure (3.11) have also been studied by a number of authors (see e.g. [30–33] and references therein).

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